Equivalence of ideal gases in two dimensions and Landen's relations

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Based on a recently given unified formulation of the statistical thermodynamics of ideal gases, an equivalence between the Bose and Fermi gases in two dimensions is established. This equivalence is shown to be a consequence of Landen's relations, found some 200 years ago, relating the Bose and Fermi fugacities through the Euler transformation. The specific heat result due to May [Phys. Rev. **135**, A1515 (1964)] follows at once. The special nature of two-dimensional solutions is described. [S1063-651X(97)14802-4]

I. INTRODUCTION

It has long been known that ideal gases at low temperatures exhibit a peculiar dependence on the number of dimensions d whether it is an even or odd number. One can trace this difference to a square-root singularity in the momentum distribution function for odd-numbered dimensions and to a log singularity for even-numbered dimensions. For this reason, the two-dimensional (2D) gases may well possess simpler properties than the 1D gases. It was in fact shown some 30 years ago by May [1] that the 2D ideal Bose and Fermi gases have the same specific heat at the same temperature. No such statement has been made for the 1D gases. Also, see McKelvey and Pulver [2].

This result of May seems to have become well known [3,4]; however, as far as we know, the underlying structure has never been examined to reveal the special nature of the solution. The work has not been extended to deduce other physical relations. This lack of progress may have been due to May's method of proof, which has perhaps obscured the essential connection. In his approach, the fugacities are eliminated in favor of some scaling temperature, which, however, does not possess the same physical significance.

In this work, we shall show that there exists a complete equivalence between the two gases if $d=2$. This equivalence is established through a unified formulation of the statistical thermodynamics of ideal gases by polylogs. To this formulation in 2D is applied certain formal relations found some 200 years ago by Landen [5], a contemporary of Euler. We shall see that the specific heat result of May follows at once. We can also see that the two gases have the same entropy. We shall learn that there exists a simple relationship between the two fugacities that underlies the equivalence. It is a relationship possible only in 2D (1D if relativistic).

II. UNIFIED STATISTICAL THERMODYNAMICS OF IDEAL GASES

It was recently shown that the statistical thermodynamics of ideal quantum gases can be unified [6]. The reduced density of an ideal gas in d dimensions ($\rho=N/V$ the number density and λ the thermal wavelength) is expressible as

$$\rho\lambda^d = \text{sgn}(\zeta)\text{Li}_{d/2}(\zeta), \quad \zeta = \begin{cases} z & \text{if Bose} \\ -z & \text{if Fermi,} \end{cases} \quad (1)$$

where z is the fugacity and $\text{Li}_m(s)$ is the polylog of m and s. If $m \geq 1$, $\text{Li}_m(s)$ is analytic everywhere in s excluding the branch cut from $s=1$ to ∞. It is real if s is real and $-\infty < s < 1$. A useful integral representation for it is [6]

$$\text{Li}_{m+1}(s) = \frac{1}{\Gamma(m+1)} \int_0^s \left[\left\langle \log \frac{s}{t} \right\rangle^m dt/(1-t) \right], \quad \text{Re}\, s < 1. \quad (2)$$

In this formulation, the domain of the Bose gas is $0<\zeta<1$, while the domain of the Fermi gas is $-\infty<\zeta<0$. From (1) one can then obtain the basic thermodynamic quantities—pressure P, energy U, and entropy S:

$$\rho^{-1}\beta P = \text{Li}_{d/2+1}(\zeta)/\text{Li}_{d/2}(\zeta), \quad (3)$$

$$\beta U/N = (d/2)\text{Li}_{d/2+1}(\zeta)/\text{Li}_{d/2}(\zeta), \quad (4)$$

$$S/Nk = (d/2+1)\text{Li}_{d/2+1}(\zeta)/\text{Li}_{d/2}(\zeta) - \log|\zeta|, \quad (5)$$

where $\beta=1/kT$ and also fluctuation quantities such as the number fluctuations Y,

$$Y = [\text{Li}_{d/2-1}(\zeta)]/[\text{Li}_{d/2}(\zeta)]. \quad (6)$$

Note that Y is related to the susceptibility by $\chi=(\beta/\rho)Y$. In (1) a kinematical factor due to the spin or polarization degeneracy is not included for simplicity. For massless systems (e.g., photon gas) the same results [Eqs. (1) and (3)–(6)] apply, provided that λ is replaced by an equivalent one and $d/2$ by d therein [6]. Observe that the above thermodynamic quantities are functions of only two parameters, the fugacity and dimensions.

The unified formulation shows that the thermodynamic properties are described by the structural properties of polylogs. There are two classes of polylogs required, those of integral and half-integral orders, respectively, for even- and odd-numbered dimensions. The polylogs of half-integral order contain a square root singularity and are still poorly understood analytically. The polylogs of integral order contain only a log singularity and are much better developed. The higher order ones (e.g., dilog, trilog), although given only as integral functions, have functional relations such as duplication and inversion. The lower-order ones, starting from the monolog down, are known in closed form. This state of the knowledge of polylogs permits a much more thorough knowledge of the statistical thermodynamics in even-

numbered dimensions (e.g., $d=0$, 2, and possibly even 4) than in odd-numbered dimensions. In the next section, we shall see this particular situation illustrated.

III. LANDEN'S RELATIONS

Landen found transformation relations for the dilog and trilog. One can easily write down similar ones for the nil log and monolog. We shall merely quote these results here and give their proofs in the Appendix.

If x is real number and $x<1$ and $y=-x/(1-x)$, then

$$\mathrm{Li}_0(x)=-\mathrm{Li}_0(y)/[1+\mathrm{Li}_0(y)], \qquad (7)$$

$$\mathrm{Li}_1(x)=-\mathrm{Li}_1(y), \qquad (8)$$

and

$$\mathrm{Li}_2(x)=-\mathrm{Li}_2(y)-\tfrac{1}{2}[\mathrm{Li}_1(y)]^2. \qquad (9)$$

The relation for the nil log (7) is artificial, merely meant to dress it in the spirit of Landen. The relation for the trilog, which is progressively more complicated, is not given here since it is not used in this work. It may be found in the book by Lewin [5]. These relations of Landen express the transformation of polylogs under the Euler transformation of x to y [7]. It may be recalled that the Euler transformation maps a line into a line and also a circle into a circle, being a bilinear transformation [8].

IV. APPLICATIONS IN TWO DIMENSIONS

We shall now set $d=2$ in the thermodynamic formulation given in Sec. II and see what special solutions may result therefrom. If the reduced densities are made the same, from Eq. (1)

$$\rho\lambda^2=\mathrm{Li}_1(z_B)=-\mathrm{Li}_1(-z_F), \qquad (10)$$

where z_B and z_F are the fugacities of the Bose and Fermi gases, respectively. If both types of particles are spinless and have the same mass and their densities are also the same, then the above condition means that their temperatures are also the same. This is a physical condition that may be satisfied if $d>0$, but not necessarily if $d=0$ [9]. It is convenient to assume that by Eq. (10), both gases are at the same temperature.

By Landen's second relation, Eq. (8), then,

$$\mathrm{Li}_1(-z_F)=\mathrm{Li}_1(-z_B/(1-z_B)). \qquad (11)$$

Hence, we conclude that the same reduced density means that

$$z_F=z_B/(1-z_B). \qquad (12)$$

The two fugacities are related by the Euler transformation. Since for a given d all the thermodynamic properties are expressible as a function of the fugacity only [see (3)–(6)], Eq. (12) means that the two 2D gases must be fundamentally related through the relationship of their fugacities.

Let us apply Eq. (12) to Eqs. (3–6). First, consider the energy for the Bose gas and apply Landen's relations (8) and (9). Then,

$$\beta U(z_B)/N=\mathrm{Li}_2(z_B)/\mathrm{Li}_1(z_B)$$

$$=\frac{\mathrm{Li}_2(-z_B/(1-z_B))+1/2\{\mathrm{Li}_1(-z_B/(1-z_B))\}^2}{\mathrm{Li}_1(-z_B/(1-z_B))}.$$

$$(13)$$

By Eq. (12), we obtain at once,

$$U(z_B)/N=U(z_F)/N+1/2\beta^{-1}\mathrm{Li}_1(-z_F)$$

$$=U(z_F)/N-1/2\beta^{-1}\rho\lambda^2. \qquad (14)$$

The second term on the right-hand side of Eq. (14) is T independent. It corresponds exactly to $1/2\varepsilon_F^{(0)}$, the ground state energy per particle of the Fermi gas in 2D, where $\varepsilon_F^{(0)}$ is the spinless Fermi energy, i.e., $\varepsilon_F^{(s)}=\varepsilon_F^{(0)}/(2s+1)$. Hence, we conclude that if the two reduced densities are the same, the energies of the Bose and Fermi gases differ by the ground state energy of the Fermi gas only. It means of course that their specific heats must be the same. This is precisely the result that May had obtained [1,12].

Since the pressure and the energy have essentially the same form, [compare Eqs. (3) and (4)], one can immediately conclude also that

$$P(z_B)=P(z_F)-P_0, \qquad (15)$$

where P_0 denotes the zero-point pressure of the spinless Fermi gas. Recall that $P_0=\tfrac{1}{2}\rho\varepsilon_F^{(0)}$ [10].

Now let us turn to the entropy (5). For the Bose gas, using (8), (9), and (12) in (5), we obtain

$$S(z_B)/Nk=2\mathrm{Li}_2(z_B)/\mathrm{Li}_1(z_B)-\log z_B$$

$$=2\mathrm{Li}_2(-z_F)/\mathrm{Li}_1(-z_F)-\log z_F$$

$$=S(z_F)/Nk, \qquad (16)$$

where we have used the identity $\log(1+x)=-\mathrm{Li}_1(-x)$. See the Appendix. Hence, the entropies of the two gases are also the same at the same reduced density.

Finally, the number fluctuations. From (6), by (7), (8), and (12),

$$Y(z_B)=\mathrm{Li}_0(z_B)/\mathrm{Li}_1(z_B)$$

$$=(1+z_F)Y(z_F). \qquad (17)$$

Thus, the number fluctuations are not equivalent at all temperatures. If $z_F\to\infty$ (i.e., $z_B\to1$), $Y(z_F)\to1/\log z_F$ but $Y(z_B)\to z_F/\log z_F$. The former vanishes while the latter diverges. But if $z_F\to0$, the two of course become equivalent, both being taken to the classical limit. The same conclusion applies to the susceptibilities since $\chi=(\beta/\rho)Y$.

V. CONCLUDING REMARKS

Our solutions can actually be seen more directly. Equation (1) implies that the grand partition function Q is [6,9,11].

$$1/V\log Q=\mathrm{sgn}(\zeta)\lambda^{-d}\mathrm{Li}_{d/2+1}(\zeta), \qquad (18)$$

where $V=L^d$, L is the length of a hypercube. Hence, if $d=2$, by Landen's relations (8) and (9),

$$1/V \log Q(z_B) = 1/V \log Q(z_F) - 1/2\rho^2\lambda^2, \qquad (19)$$

where now $V = L^2$. The second term on the right-hand side is linear in β. According to the grand ensemble theory [12], $U = -\partial/\partial\beta \log Q|_z$. Hence, the two energies differ by the constant accompanying the β above. The two specific heats must be the same. Also, $PV = \beta^{-1} \log Q|_\mu$, meaning that the two pressures differ by a constant. Since $S/k = -\beta^2(\partial/\partial\beta)PV|_\mu$, the two entropies must also be the same [13].

The equivalence thus arises from the fact that if $d=2$, the energy and pressure of the Bose gas are those of the Fermi gas shifted by the zero-point constants of the Fermi gas. It is unique to two dimensions. It cannot occur in any other dimension. We shall briefly examine why it might be precluded in other dimensions.

Consider $d=4$, the next simplest even numbered dimension. Applying it to Eq. (1) and using Landen's relation (9), we obtain

$$\begin{aligned} -\mathrm{Li}_2(-z_F) &= \mathrm{Li}_2(z_B) \\ &= -\mathrm{Li}_2(-z_B/(1-z_B)) \\ &\quad -\tfrac{1}{2}\{\mathrm{Li}_1(-z_B/(1-z_B))\}^2. \end{aligned} \qquad (20)$$

Thus, $z_F \neq z_B/(1-z_B)$. The relationship is more complicated, although it probably contains the Euler form in some ways. Let us next consider $d=1$, the simplest odd-numbered dimension. Then,

$$-\mathrm{Li}_{1/2}(-z_F) = \mathrm{Li}_{1/2}(z_B). \qquad (21)$$

There are no Landen's relations for the polylogs of half-integral order. The relationship between the two fugacities cannot be exactly determined, although one again suspects that the Euler form might also be involved.

The relationships between the two fugacities cannot be the Euler form exactly if $d \neq 2$. Without this knowledge, it is not possible to establish any connection between the thermodynamics of the Bose and Fermi gases. It seems reasonable to conclude that the equivalence found in 2D is unique, not to be found in any other physical dimension. If one were to consider these gases in the relativistic limit, the equivalence would be found in 1D.

ACKNOWLEDGMENTS

APPENDIX: LANDEN'S RELATIONS

We shall prove Landen's relations given in Sec. III. The proofs for the first two relations are trivial since $\mathrm{Li}_0(x) = x/(1-x)$ and $\mathrm{Li}_1(x) = -\log(1-x)$, where $x < 1$. One can obtain $\mathrm{Li}_1(x)$ from Eq. (2) directly and $\mathrm{Li}_0(x)$ from $\mathrm{Li}_1(x)$ by the recurrence relation $\mathrm{Li}_m(x) = (xd/dx)\mathrm{Li}_{m+1}(x)$ setting $m=0$. Also, see [6,9].

To prove the third relation (9), we use the integral representation (2). Let $t = -x/(1-x)$. Then,

$$\mathrm{Li}_2(s) = -\int_0^\sigma [\log s - \log\{-x/(1-x)\}]dx/(1-x), \qquad (A1)$$

where $\sigma = -s/(1-s)$. Then,

$$\begin{aligned} \mathrm{Li}_2(s) &= -\int_0^\sigma \log\{\sigma/x\}dx/(1-x) \\ &\quad + \int_0^\sigma \log\{(1-\sigma)/(1-x)\}dx/(1-x) \\ &= -\mathrm{Li}_2(\sigma) - \tfrac{1}{2}(\mathrm{Li}_1(\sigma))^2 \quad \text{Q.E.D.} \end{aligned} \qquad (A2)$$

In the last step we have used the monolog form $\mathrm{Li}_1(x) = -\log(1-x)$, $x < 1$. For the original proofs of the above and for the trilog, see Lewin's book [5].

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[10] This result does not appear to have been established before, although it could have been deduced using May's work. It would follow from the thermodynamic potential Eq. (5.2.27), given in Dash's book (Ref. [3]), or more simply, from $PV = k^{-1}U$, where $k = d/2$ or $d, d > 0$, depending on whether a nonrelativistic or relativistic ideal gas.
[11] There is a typographical error in the grand partition function given in Ref. [9]. A minus sign before d is missing.
[12] R. K. Pathria, *Statistical Mechanics*, 2nd ed. (Butterworth-Heinenmann, Oxford, 1996), p. 230. (Also see p. 253 of the first edition.)
[13] If the spin degeneracy is to be taken into account, one must restore $(2s+1)$ in the expression for the density, (1). The other expressions (3)–(5) are not affected, since they are given as "per particle or per density." If we assume that T and λ are the same, the condition (10) results in the right-hand side of (15) being multiplied by ρ_B/ρ_F, but leaves (14) and (16) unaffected, provided that $\varepsilon_F^{(0)}$ is replaced by $\varepsilon_F^{(s)}$.